Topic 4 –

Separable first order ODEs

Def: A first order ODE is called <u>separable</u> if it is of the form

$$\underbrace{N(y)\cdot y'}_{\text{y's on one side}} = \underbrace{M(x)}_{\text{x's on other side}}$$

Ex:

$$\underbrace{y^2}_{N(y)}\underbrace{\frac{dy}{dx}}_{\cdot\, y'} = \underbrace{x-5}_{M(x)}$$

$\boxed{\text{non-linear ODE order 1}}$

Ex:

$$y' = \frac{x^2}{y}$$

$\boxed{\text{non-linear ODE order 1}}$

which becomes

$$\underbrace{y}_{N(y)}\cdot\underbrace{y'}_{\frac{dy}{dx}} = \underbrace{x^2}_{M(x)}$$

How to solve a separable ODE

Formal notation	Informal notation
$N(y)\cdot y' = M(x)$	$N(y)\cdot \frac{dy}{dx} = M(x)$
$\downarrow$	$\downarrow$
$N(y(x))\cdot y'(x) = M(x)$	$N(y)\,dy = M(x)\,dx$ [informal differential form notation]
$\downarrow$	$\boxed{\int N(y)\,dy = \int M(x)\,dx}$ Now integrate.
$\int N(y(x))\cdot y'(x)\,dx = \int M(x)\,dx$	
$\downarrow$ Sub: $u = y(x)$, $du = y'(x)\,dx$	
$\boxed{\int N(u)\,du = \int M(x)\,dx}$ Now integrate. Here $u = y$.	

<u>Ex:</u> Solve

$$y^2 \frac{dy}{dx} = x - 5$$

non-linear ODE order 1

Also where does this solution exist?

Formal	Informal
$y^2(x) \cdot y'(x) = x-5$	$y^2 \frac{dy}{dx} = x-5$
$\int y^2(x) \cdot y'(x)\, dx = \int (x-5)\, dx$	$y^2 dy = (x-5) dx$
$u = y(x)$, $du = y'(x) dx$	
$\int u^2(x) dx = \int (x-5) dx$	$\int y^2 dy = \int (x-5) dx$
$\frac{u^3}{3} = \frac{x^2}{2} - 5x + C$	$\frac{y^3}{3} = \frac{x^2}{2} - 5x + C$
where C is a constant	where C is a constant

$\boxed{u = y(x)}$

$$y^3 = \frac{3}{2}x^2 - 15x + D$$

where D is a constant

$$y^3 = \frac{3}{2}x^2 - 15x + D$$

where D is a constant

Thus, $y^3 = \frac{3}{2}x^2 - 15x + D$ gives a solution to $y^2 \frac{dy}{dx} = x - 5$

You can solve for y to get

$$y = \left(\frac{3}{2}x^2 - 15x + D\right)^{1/3}$$

can solve for y since 3 is odd

is a solution on $(-\infty, \infty)$

$\square$

Ex: Solve

$$\frac{dy}{dx} + 2xy = 0$$

Where does the solution exist?

Informal method

$$\frac{dy}{dx} = -2xy$$

$$\frac{dy}{y} = -2x\,dx$$

$$\int \frac{dy}{y} = -\int 2x\,dx$$

$$\ln|y| = -x^2 + C_1$$

$$|y| = e^{-x^2+C_1}$$

$$|y| = e^{-x^2} \cdot e^{C_1}$$

$C_2 = e^{C_1}$

$$|y| = C_2 e^{-x^2}$$

$C = \pm C_2$

$$y = C e^{-x^2}$$

Let's test this solution:

$$y = ce^{-x^2}$$

$$\frac{dy}{dx} = ce^{-x^2} \cdot (-2x) = -2cxe^{-x^2}$$

So,

$$\frac{dy}{dx} + 2xy = -2cxe^{-x^2} + 2xce^{-x^2} = 0$$

Thus, $y = ce^{-x^2}$ does in fact solve $\frac{dy}{dx} + 2xy = 0$

It does so on $I = (-\infty, \infty)$.

Suppose we wanted a solution to

$$\boxed{\begin{aligned} &\frac{dy}{dx} + 2xy = 0 \\ &y(1) = 2 \end{aligned}} \quad (*)$$

Plugging $y(1) = 2$ into $y = ce^{-x^2}$ gives

$$2 = y(1) = ce^{-(1)^2}$$

So, $c = 2e$.

Thus, $y = 2ee^{-x^2} = 2e^{-x^2+1}$ is a solution to $(*)$.