

Topic 4 -

Separable first order

ODEs



Def: A first order ODE
is called separable if it
is of the form

$$\underbrace{N(y) \cdot y'}_{y\text{'s on one side}} = \underbrace{M(x)}_{x\text{'s on other side}}$$

Ex: $y^2 \frac{dy}{dx} = x - 5$

$\underbrace{y^2}_{N(y)} \cdot \underbrace{\frac{dy}{dx}}_{y'}$ $\underbrace{x - 5}_{M(x)}$

non-linear
ODE
order 1

Ex: $y' = \frac{x^2}{y}$

non-linear
ODE
order 1

which becomes

$$\underbrace{y}_{N(y)} \cdot \underbrace{y'}_{\frac{dy}{dx}} = \underbrace{x^2}_{M(x)}$$

How to solve a separable ODE

Formal notation

$$N(y) \cdot y' = M(x)$$



$$N(y(x)) \cdot y'(x) = M(x)$$



$$\int N(y(x)) \cdot y'(x) dx = \int M(x) dx$$

↓

Sub;
 $u = y(x)$
 $du = y'(x) dx$

$$\int N(u) du = \int M(x) dx$$

Now integrate
Here $u = y$.

Informal notation

$$N(y) \cdot \frac{dy}{dx} = M(x)$$



$$N(y) dy = M(x) dx$$

[informal differential form notation]

$$\int N(y) dy = \int M(x) dx$$

Now integrate.

Ex: Solve

$$y^2 \frac{dy}{dx} = x - 5$$

non-linear
ODE
order 1

Also where does this solution exist?

Formal

$$y^2(x) \cdot y'(x) = x - 5$$

$$\int y^2(x) \cdot y'(x) dx = \int (x-5) dx$$

$$u = y(x) \\ du = y'(x) dx$$

$$\int u^2(x) dx = \int (x-5) dx$$

$$\frac{u^3}{3} = \frac{x^2}{2} - 5x + C$$

where C is a constant

Informal

$$y^2 \frac{dy}{dx} = x - 5$$

$$y^2 dy = (x-5) dx$$

$$\int y^2 dy = \int (x-5) dx$$

$$\frac{y^3}{3} = \frac{x^2}{2} - 5x + C$$

where C is a constant

$$u = y(x)$$

$$y^3 = \frac{3}{2}x^2 - 15x + D$$

where D is a constant

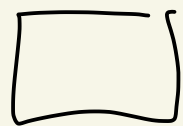
$$y^3 = \frac{3}{2}x^2 - 15x + D$$

where D is a constant

Thus, $y^3 = \frac{3}{2}x^2 - 15x + D$ gives a solution to $y^2 \frac{dy}{dx} = x - 5$

You can solve for y to get $y = \left(\frac{3}{2}x^2 - 15x + D\right)^{1/3}$ can solve for y since 3 is odd

on $(-\infty, \infty)$ is a solution



Ex: Solve

$$\frac{dy}{dx} + 2xy = 0$$

Where does the solution exist?

Informal method

$$\frac{dy}{dx} = -2xy$$

$$\frac{dy}{y} = -2x dx$$

$$\int \frac{dy}{y} = -\int 2x dx$$

$$\ln|y| = -x^2 + C_1$$

$$|y| = e^{-x^2 + C_1}$$

$$|y| = e^{-x^2} \cdot e^{C_1}$$

$$|y| = c_2 e^{-x^2}$$

$$y = c e^{-x^2}$$

$$c_2 = e^{C_1}$$

$$c = \pm c_2$$

Let's test this solution:

$$y = ce^{-x^2}$$

$$\frac{dy}{dx} = ce^{-x^2} \cdot (-2x) = -2cxe^{-x^2}$$

So,

$$\frac{dy}{dx} + 2xy = -2cxe^{-x^2} + 2cxe^{-x^2} = 0$$

Thus, $y = ce^{-x^2}$ does in fact solve $\frac{dy}{dx} + 2xy = 0$

It does so on $I = (-\infty, \infty)$.

Suppose we wanted a solution to

$$\boxed{\begin{aligned} \frac{dy}{dx} + 2xy &= 0 \\ y(1) &= 2 \end{aligned}} \quad (*)$$

Plugging $y(1) = 2$ into $y = ce^{-x^2}$ gives

$$2 = y(1) = ce^{-(1)^2}$$

So, $c = 2e$.

Thus, $y = 2ee^{-x^2} = 2e^{-x^2+1}$ is a solution to $(*)$.